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# The Weierstrass Transform for a Class of Generalized Functions

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The classical theory of the Weierstrass transform is extended to a generalized function space which is the dual of a testing function space consisting of purely entire functions with certain growth conditions developed by Kenneth B. Howell. An inversion formula and characterizations for this transform are obtained. A comparative study with the existing literature is also undertaken. © 1998 Academic Press

The conventional Weierstrass transform of a suitably restricted function $f(t)$ on the real axis $\mathbb{R}$ is defined as

$$\frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} f(t) e^{-(z-t)^2/4} \, dt,$$

where $z$ is a complex variable (see, for example, [3]). This transform arises naturally in problems involving the heat equation for one dimensional flow.

Zemanian [12] defined and investigated the Weierstrass transform of a certain class of generalized functions which are duals of the so-called testing function spaces $\mathscr{W}_{a,b}$ and $\mathscr{W}(\alpha, \beta)$. The inversion formulas were also obtained. The intrinsic connection between the Weierstrass and the Laplace transforms is also brought out in Theorem 7.2.1 [12, p. 208], Theorem 7.2.2 [12, p. 209], and Theorem 7.3.1 [12, p. 211]. Further, necessary and sufficient conditions for a function $F(z)$ to be the Weierstrass transform of a generalized function are also obtained (see [12,

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Theorem 7.3.5, p. 213]). On the other hand the Weierstrass transform of bounded functions, $\mathscr{L}^p$-functions, and certain other functions with prescribed growth conditions are all characterized by Hirshman and Widder [3]. For other types of Weierstrass transforms on single or multidimensional spaces we refer to [2].

We first observe that the class $\mathscr{S}'$ of tempered distributions is contained in $\mathscr{W}'(-2\beta, -2\alpha)$. Therefore the Weierstrass transform theory can be made applicable to $\mathscr{S}'$. The theory of Laplace transform can be easily applied to $\mathscr{S}'$ and in fact it is generalized to the space $\mathscr{L}'(\alpha, \beta)$ (when $\alpha < \beta$) defined by Zemanian [12]. However, there is only an isomorphism between $\mathscr{W}'(-2\beta, -2\alpha)$ and $\mathscr{L}'(\alpha, \beta)$. In this sense the Weierstrass transform is not directly applicable to the generalized function space $\mathscr{L}'(\alpha, \beta)$ when $\alpha < \beta$. Also no such theory is available for $\mathscr{W}'(-2\beta, -2\alpha)$ and $\mathscr{L}'(\alpha, \beta)$ when $\alpha > \beta$.

In this paper we start with the testing function space $\mathscr{G}$ and the generalized function space $\mathscr{G}'$, the dual of $\mathscr{G}$, introduced and developed in a sequence of papers by Kenneth B. Howell [4–8]. For suitable real numbers $\alpha$ and $\beta$ we shall show that $\mathscr{L}'(\alpha, \beta)$ can be identified as a sub-space of $\mathscr{G}'$. We introduce the Weierstrass transform on $\mathscr{G}'$, obtain an inversion formula, and also characterize the Weierstrass transform of elements from both $\mathscr{G}$ and $\mathscr{G}'$.

In contrast to the generalized function spaces which are duals of testing function spaces consisting of smooth complex valued functions of a single real variable, the space $\mathscr{G}'$ is the dual of the testing function space $\mathscr{G}$ which consists of entire functions with certain growth conditions. One may observe that the Fourier transform theory is applicable to all of $\mathscr{G}'$ (see [4–8]), and the convolution transform theory with kernel $(1/2)\operatorname{sech}(z/2)$ is also applicable to a certain sub-space $\mathbb{E}'_d$ ($\mathscr{S}' \subset \mathbb{E}'_d$) of $\mathscr{G}'$ (see [10]).

## 1. PRELIMINARIES

In this section we shall state the concepts and results which will be needed in the sequel. For details and proofs we refer to [4–8].

DEFINITION 1.1. The space $\mathscr{G}$ consists of all entire functions $\phi(z)$ of one complex variable $z$ such that for every $\alpha > 0$

$$\|\phi\|_\alpha = \sup_{z \in B_\alpha} e^{\alpha |\operatorname{Re} z|} |\phi(z)| < \infty,$$

where $B_\alpha = \{x + iy = z \in \mathbb{C} : |y| \leq \alpha\}$.

DEFINITION 1.2. The space $\mathscr{G}^c$ consists of all entire functions $\phi(z)$ of one complex variable $z$ such that for every $\alpha > 0$ there is a corresponding $\gamma \geq 0$ such that

$$\sup_{z \in B_\alpha} e^{-\gamma|\mathrm{Re}\, z|} |\phi(z)| < \infty.$$

$\mathscr{G}$ can be given a Frechét topology with the multi-norm $\{\|\phi\|_\alpha\}_{\alpha \geq 0}$, i.e., $\phi_n \to \phi$ in $\mathscr{G} \Leftrightarrow \|\phi_n - \phi\|_\alpha \to 0$ as $n \to \infty$ $\forall\, \alpha > 0$.

For each $\phi \in \mathscr{G}$ its Fourier transform is defined as

$$\mathscr{F}(\phi)(z) = \hat{\phi}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(t) e^{-itz}\, dt.$$

THEOREM 1.3. *The Fourier transform $\mathscr{F}$ is a continuous, linear, one-to-one mapping of $\mathscr{G}$ onto $\mathscr{G}$ with a continuous inverse. Moreover, for every $\alpha > 0$, $\beta > 0$, and $\phi \in \mathscr{G}$,*

$$\|\mathscr{F}(\phi)\|_\alpha \leq \left( \frac{1}{\beta} \frac{\sqrt{2}}{\sqrt{\pi}} \right) \|\phi\|_{\alpha+\beta}.$$

THEOREM 1.4. *For each $\psi \in \mathscr{G}$, the mapping $\phi \to \phi * \psi$ is a continuous linear mapping from $\mathscr{G}$ into $\mathscr{G}$ where*

$$(\phi * \psi)(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(t) \psi(z-t)\, dt$$

*with $\|\phi * \psi\|_\alpha \leq (2/\beta\sqrt{2\pi}) \|\phi\|_\alpha \|\psi\|_{\alpha+\beta}$ for any $\alpha > 0$, $\beta > 0$. Moreover, the convolution product on $\mathscr{G}$ is commutative and associative and for any $\phi$, $\psi \in \mathscr{G}$, $m \in \mathbb{N}$ we have*

(i) $(\phi * \psi)^{(m)} = \phi * \psi^{(m)} = \phi^{(m)} * \psi$

(ii) $\mathscr{F}(\phi * \psi) = \mathscr{F}(\phi)\mathscr{F}(\psi)$ *and* $\mathscr{F}(\phi\psi) = \mathscr{F}(\phi) * \mathscr{F}(\psi)$.

Let us denote the space of continuous linear functionals on $\mathscr{G}$ by $\mathscr{G}'$. $\mathscr{G}'$ will be given the weak* topology (see [11, 3.14, p. 66]). This is a locally convex vector topology on $\mathscr{G}'$ and compact sets of $\mathscr{G}'$ in this topology will be called Weak* compact sets. Moreover $f_n \to f$ in the weak* topology of $\mathscr{G}'$ if and only if $f_n(\phi) \to f(\phi)$ for every $\phi \in \mathscr{G}$.

THEOREM 1.5. *For each $f \in \mathscr{G}'$ there are finite positive constants $C$ and $\alpha$ such that $|\langle f, \phi \rangle| \leq C\|\phi\|_\alpha$ for all $\phi \in \mathscr{G}$.*

The Fourier transform $\mathscr{F}[f]$ of an $f \in \mathscr{G}'$ is defined to be an element of $\mathscr{G}'$ by $\langle \mathscr{F}[f], \phi \rangle = \langle f, \mathscr{F}(\phi) \rangle$ for $\phi \in \mathscr{G}$.

THEOREM 1.6. *$\mathscr{F}$ is continuous, linear, one-to-one mapping from $\mathscr{G}'$ onto $\mathscr{G}'$ with continuous inverse.*

If $\Gamma \in \mathscr{G}'$, $\psi \in \mathscr{G}$ the convolution of $\Gamma$ with $\psi$, $\Gamma * \psi$, is the function given by $(\Gamma * \phi)(z) = \mathscr{F}(\tau_z \check{\psi})$ where $\tau_z \psi(w) = \check{\psi}(z-w)$ with $\check{\psi}(z) = \psi(-z)$.

THEOREM 1.7. *If $\Gamma \in \mathscr{G}'$, $\psi \in \mathscr{G}$ then $\Gamma * \psi \in \mathscr{G}^c$. Moreover,*

(i) $(\Gamma * \phi) * \psi = \Gamma * (\phi * \psi)$

(ii) $\mathscr{F}(\Gamma * \psi) = \mathscr{F}(\Gamma)\mathscr{F}(\psi)$ *and* $\mathscr{F}(\psi\Gamma) = \mathscr{F}(\Gamma) * \mathscr{F}(\psi)$, *where* $(\psi\Gamma)(\phi) = \Gamma(\psi\phi)$ *for* $\phi \in \mathscr{G}$.

## 2. THE WEIERSTRASS TRANSFORM

DEFINITION 2.1. If $f \in \mathscr{G}'$ then the Weierstrass transform $\tilde{f}$ of $f$ is defined to be

$$\tilde{f}(w) = \langle f(z), k(w-z, 1)\rangle = \sqrt{2\pi}\,(f * k_1)(w) \qquad (w \in \mathbb{C}),$$

where $k(\zeta, t) = k_t(\zeta) = (1/\sqrt{4\pi t})\mathrm{e}^{-\zeta^2/4t}$ for $t > 0$, $\zeta \in \mathbb{C}$ (One can easily verify that $k(w-z, 1)$, as a function of $z$, belongs to $\mathscr{G}$ for every $w \in \mathbb{C}$).

The following proposition is an immediate consequence of Theorem 1.7;

PROPOSITION 2.2. *If $f \in \mathscr{G}'$ then its Weierstrass transform $\tilde{f}$ belongs to $\mathscr{G}^c$.*

We shall now obtain the inversion formula for the Weierstrass transform of elements in $\mathscr{G}^c$ and using this, we shall also deduce the inversion formula for the Weierstrass transform of elements in $\mathscr{G}'$. Though the classical proof does not seem to be adaptable to this case, we would like to remark that the proof essentially makes use of the continuity of the Fourier transform and its inverse from $\mathscr{G}$ onto $\mathscr{G}$.

THEOREM 2.3 (Inversion for the Weierstrass transform of elements in $\mathscr{G}^c$). *If $\tilde{\phi}$ is the Weierstrass transform of $\phi \in \mathscr{G}^c$ then for $z \in \mathbb{C}$, $\phi(z) = \lim_{t \to 1-} \int_{\mathbb{R}} k(\xi + iz, t)\tilde{\phi}(i\xi)\, d\xi$.*

*Proof.* Since $\phi \in \mathscr{G}^c$ there are positive constants, $M$ and $\alpha$, such that $|\phi(\xi)| \le Me^{\alpha|\xi|}$ for all $\xi \in \mathbb{R}$. By a straightforward calculation we obtain

$$\begin{aligned}|\tilde{\phi}(x+iy)| &= |\langle \phi(\xi), k(x+iy-\xi, 1)\rangle| \\ &= \frac{1}{\sqrt{4\pi}}\left|\int_{\mathbb{R}} e^{-(x+iy-\xi)^2/4}\phi(\xi)\, d\xi\right| \\ &\le e^{y^2/4}e^{\alpha|x|}N,\end{aligned}$$

where $N = (1/\sqrt{4\pi})\int_{\mathbb{R}} e^{-s^2/4} M e^{\alpha|s|}\,ds$ which is clearly finite. From this it immediately follows that $\int_{\mathbb{R}} k(\xi + iz, t)\tilde{\phi}(i\xi)\,d\xi$ exists whenever $0 < t < 1$. To prove the inversion formula, let

$$\psi(z,t) = \int_{\mathbb{R}} k(\xi + iz, t)\,\tilde{\phi}(i\xi)\,d\xi.$$

We need to show that, as $t \to 1-$, $\psi(z,t) \to \phi(z)$ for all $z \in \mathbb{C}$. Writing $\psi$ in all detail,

$$\begin{aligned}\psi(z,t) &= \int_{\mathbb{R}} k(\xi + iz, t)\,\tilde{\phi}(i\xi)\,d\xi \\ &= \frac{1}{4\pi\sqrt{t}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(\xi+iz)^2/4t} e^{-(i\xi - s)^2/4}\phi(s)\,ds\,d\xi.\end{aligned}$$

It is not hard to show that, so long as $0 < t < 1$, the above integrand is an analytic function of both $s$ and $\xi$ as well as being an absolutely integrable function on $\mathbb{R}^2$. This allows us to use the "change of variables" (actually, a change of variables and Cauchy's theorem) and the interchanging of the order of integration employed in the next set of computations:

$$\begin{aligned}\psi(z,t) &= \frac{1}{4\pi\sqrt{t}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(\xi+iz)^2/4t} e^{-(i\xi - s)^2/4}\phi(s)\,ds\,d\xi \\ &= \frac{1}{4\pi\sqrt{t}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-w^2/4t} e^{-(iw - v)^2/4}\phi(v + z)\,dv\,dw.\end{aligned}$$

This last integral is equal to

$$\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} \phi(v + z) e^{-v^2/4}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\left[-\left(\frac{1-t}{t}\right)\sigma^2\right] e^{iv\sigma}\right) d\sigma\,dv.$$

Noting that the inner integral on the last line is the inverse Fourier transform of a Gaussian, we easily obtain

$$\psi(z,t) = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} \phi(v + z) e^{-v^2/4}\eta_\gamma(v)\,dv = \frac{1}{\sqrt{t}}(\Gamma * \eta_\gamma)(0),$$

where

$$\Gamma(v) = \phi(v + z)e^{-v^2/4}, \qquad \eta_\gamma(v) = \frac{1}{\sqrt{2}}\gamma e^{-(\gamma v)^2/4}, \qquad \text{and}$$

$$\gamma = \sqrt{\frac{t}{1-t}}\,.$$

By straightforward analysis or simply appealing to, say, [8, Lemma 2.3, p. 570], we immediately obtain

$$\lim_{t\to 1-} \psi(z,t) = \phi(0+z)e^{-0^2/4} = \phi(z). \quad \blacksquare$$

The following Lemma 2.4 and Corollary 2.5 can be seen to be particular cases of Lemma 2.3 in [8, p. 570] taking $\eta(z) = (1/\sqrt{2})e^{-z^2/4}$ and $\gamma = (1/\sqrt{t})$.

LEMMA 2.4. *If $k_t(\zeta) = k(\zeta, t)$ and if $\phi \in \mathscr{G}$ then*

$$\sqrt{2\pi}(\phi * k_t) \to \phi \qquad \text{in } \mathscr{G} \text{ as } t \to 0^+.$$

COROLLARY 2.5. *If $f \in \mathscr{G}'$ then $\sqrt{2\pi}(f * k_t) \to f$ in $\mathscr{G}'$ as $t \to 0^+$.*

THEOREM 2.6 (Inversion Theorem). *If $f \in \mathscr{G}'$ and if $\tilde{f}$ is the Weierstrass transform of $f$ then*

$$\langle f, \phi\rangle = \sqrt{2\pi}\lim_{r\to 0}\left\langle \lim_{t\to 1-}\int_{\mathbb{R}} k(y+ix,t)\big(\tilde{f} * k_r\big)(iy)\,dy, \phi(x)\right\rangle.$$

*Proof.* We observe that using properties of convolution

$$\tilde{f} * k_r = \big(\sqrt{2\pi} f * k_1\big) * k_r = \sqrt{2\pi}(f * k_r) * k_1 = (f * k_r)^{\sim}.$$

Now $(f * k_r)^{\sim}$ being the Weierstrass transform of $f * k_r$ belongs to $\mathscr{G}^c$ by Theorem 1.7. Applying Theorem 2.3 we get

$$\begin{aligned}(f * k_r)(x) &= \lim_{t\to 1-}\int_{\mathbb{R}} k(y+ix,t)(f * k_r)^{\sim}(iy)\,dy \\ &= \lim_{t\to 1-}\int_{\mathbb{R}} k(y+ix,t)\big(\tilde{f} * k_r\big)(iy)\,dy.\end{aligned}$$

Thus

$$\begin{aligned}\langle f, \phi\rangle &= \sqrt{2\pi}\lim_{r\to 0}\langle (f * k_r), \phi\rangle \qquad \text{(by Corollary 2.5)} \\ &= \sqrt{2\pi}\lim_{r\to 0}\left\langle \lim_{t\to 1-}\int_{\mathbb{R}} k(y+ix,t)\big(\tilde{f} * k_r\big)(iy)\,dy, \phi(x)\right\rangle.\end{aligned}$$

COROLLARY 2.7. *If $f, g \in \mathscr{G}'$ and if their respective Weierstrass transforms $\tilde{f}$ and $\tilde{g}$ are equal pointwise, then $f = g$ in the sense of equality in $\mathscr{G}'$.*

*Proof.* For $\phi \in \mathscr{G}$ by Theorem 2.6

$$\begin{aligned}\langle f, \phi\rangle &= \sqrt{2\pi} \lim_{r\to 0} \left\langle \lim_{t\to 1-} \int_{\mathbb{R}} k(y+ix,t)\big(\tilde{f} * k_r\big)(iy)\,dy, \phi(x)\right\rangle \\ &= \sqrt{2\pi} \lim_{r\to 0} \left\langle \lim_{t\to 1-} \int_{\mathbb{R}} k(y+ix,t)(\tilde{g} * k_r)(iy)\,dy, \phi(x)\right\rangle \\ &= \langle g, \phi\rangle. \quad \blacksquare\end{aligned}$$

## 3. CHARACTERIZATIONS OF THE WEIERSTRASS TRANSFORM

In this section we shall characterize the Weierstrass transform of elements of $\mathscr{G}$ and $\mathscr{G}'$. The characterization for Weierstrass transforms of elements of $\mathscr{G}$ is done in the canonical way. However, in the case of $\mathscr{G}'$ the proof of the theorem illustrates the effective use of the Fourier transform theory on $\mathscr{G}'$ (developed in [4–8] and other new concepts and results regarding $\mathscr{G}'$ (developed in [9]).

THEOREM 3.1 (Characterization of the Weierstrass transform for elements of $\mathscr{G}$). *The conditions*

(I) *$F(w)$ is entire and $|F(u+iv)| \le M e^{v^2/4}$*

(II) $\|e^{-tD^2}F\|_\alpha \le M_\alpha$ $(0 < t < 1, \alpha > 0)$,

*where, using the notation of* [3, *p*. 177],

$$e^{-tD^2}F(w) = \int_{\mathbb{R}} k(\xi + iw, t) F(i\xi)\,d\xi,$$

*are necessary and sufficient conditions that $F(w) = \tilde{\phi}(w)$ for some $\phi \in \mathscr{G}$.*

*Proof.* Let us first prove that (I) and (II) are necessary. If $F(w) = \tilde{\phi}(w)$ for some $\phi \in \mathscr{G}$, then it is easily checked that $F$ satisfies (I) with $w = u + iv$ and $|\phi(\xi)| \le M$. As a simple application of Morera's theorem one can show that $e^{-tD^2}F(w)$ is entire. Moreover by Fubini's theorem and a series of straightforward calculations,

$$e^{-tD^2}F(u) = \int_{\mathbb{R}} k(\xi + iu, t) F(i\xi)\,d\xi = \int_{\mathbb{R}} k(u-\eta, 1-t)\phi(\eta)\,d\eta.$$

In obtaining the last equality we have used Theorem 2.4 in [3, Chap. VIII, p. 177]. Therefore

$$e^{-tD^2}F(u) = \int_{\mathbb{R}} k(u-\eta, 1-t)\phi(\eta)\,d\eta$$

for real $u$. Furthermore since both sides are entire (the last integral is in fact a convolution of two elements of $\mathscr{G}$ and thus is an element of $\mathscr{G}$) $e^{-tD^2}F(w) = \sqrt{2\pi}(\phi * k_{1-t})(w)$ for all $w \in \mathbb{C}$ by the principle of analytic continuation. By Lemma 2.4, $\sqrt{2\pi}(\phi * k_{1-t})(w)$ converges in $\mathscr{G}$ to $\phi$ as $t \to 1-$. Thus for each $\alpha > 0$ and $\varepsilon > 0$ there exists $c < 1$ such that $\|e^{-tD^2}F - \phi\|_\alpha < \varepsilon$ ($\forall\ t > c$). Thus

$$\|e^{-tD^2}F\|_\alpha \le \varepsilon + \|\phi\|_\alpha \qquad \forall\ t > c. \tag{1}$$

For $t \le c$ we observe that as $1 - t < 1$

$$\begin{aligned}\left|e^{\alpha|u|}e^{-tD^2}F(w)\right| &\le \frac{e^{\alpha|u|}}{\sqrt{4\pi(1-t)}}\int_{\mathbb{R}}\left|\bar{e}^{(w-\xi)^2/4(1-t)}\phi(\xi)\right|d\xi\\ &\le M\frac{e^{\alpha|u|}}{\sqrt{4\pi(1-c)}}\int_{\mathbb{R}}\bar{e}^{(u-\xi)^2/4}|\phi(\xi)|\,d\xi\\ &\le M\frac{e^{\alpha|u|}}{\sqrt{4\pi(1-c)}}\int_{\mathbb{R}}C_1e^{-\alpha|u-\xi|}C_2e^{-\beta|\xi|}\,d\xi\end{aligned}$$

(for some $\beta > \alpha$ and where $C_1 = \|e^{-z^2/4}\|_\alpha$ and $C_2 = \|\phi\|_\beta$)

$$\le M\frac{e^{\alpha|u|}}{\sqrt{4\pi(1-c)}}\int_{\mathbb{R}}e^{-\alpha|u|}e^{\alpha|\xi|}e^{-\beta|\xi|}\,d\xi = N_\alpha.$$

Thus

$$\|e^{-tD^2}F\|_\alpha \le N_\alpha \qquad \forall\ t \le c. \tag{2}$$

From (1) and (2) we obtain (II).

Conversely suppose that (I) and (II) hold. By condition (I), condition (II) (observe that condition (II) in particular implies that $|e^{-tD^2}F(\xi)|$ is uniformly bounded for $\xi \in \mathbb{R}$ and $0 < t < 1$), Theorem 3.4 of [3, Chap. VIII, p. 182] and Lemma 6.2 of [3, Chap. VIII, p. 180], we have

$$u(\sigma, t+\delta) = \int_{\mathbb{R}}k(\sigma-\xi, t)u(\xi, \delta)\,d\xi \tag{3}$$

for $0 < \delta < 1$, $0 < t < 1 - \delta$, and $-\infty < x < \infty$ where $e^{-(1-t)D^2}F(\zeta)$ is denoted by $u(\zeta, t)$.

The family $\{u(\xi, \delta)\}_\delta$ is uniformly bounded on every compact subset of $\mathbb{C}$ (in fact on every strip around the real axis by (II)) and thus is normal with respect to $\mathbb{C}$ and as such there exists a subsequence $\{u(\xi, \delta_n)\}$

converging uniformly on compact subsets to a holomorphic function $\phi(z)$ (see [1, Definition 2, Sect. 5.1, p. 220, and Theorem 15, Sect. 5.4, p. 224]). This $\phi$ is entire and (as a simple application of (II)) can be easily verified to be in $\mathscr{G}$.

Allowing $\delta \to 0$ in (3) we get by the Lebesgue dominated convergence theorem $u(\sigma, t) = \int_{\mathbb{R}} k(\sigma - \xi, t)\phi(\xi) d\xi$.

The last integral can be verified to be a continuous function of $t$, and letting $t \to 1-$ in the last equality, we see that $u(\sigma, 1-) = \int_{\mathbb{R}} k(\sigma - \xi, 1)\phi(\xi)\, d\xi$ and as in the second part of Theorem 6.3 of [3, Chap. VIII, p. 187], $u(\sigma, 1-) = F(\sigma)$ and by principle of analytic continuation $u(w, 1-) = F(w)$ or $F(w) = \tilde{\phi}(w)$ for all $w \in \mathbb{C}$. ∎

To characterize the Weierstrass transform of elements of $\mathscr{G}'$, we shall need some concepts and lemmas (from Definition 3.2 to Lemma 3.4) due to Karunakaran and Kalpakam [9].

DEFINITION 3.2. The class $\Delta$, by definition, consists of all sequences $\{\delta_n\}$ from $\mathscr{G}$ such that

$$\int_{\mathbb{R}} \delta_n(x)\, dx = \sqrt{2\pi} \tag{$\Delta1$}$$

$$\int_{\mathbb{R}} |\delta_n(x)|\, dx \le M \tag{$\Delta2$}$$

$$\lim_{n\to\infty} \int_{|x|\ge\varepsilon} (e^{\alpha|x|} - 1)|\delta_n(x)|\, dx = 0, \qquad \forall\, \alpha > 0 \text{ and } \forall\, \varepsilon > 0 \tag{$\Delta3$}$$

and the class $\hat{\Delta}$ consists of all $\{\hat{\delta}_n\}$ where $\{\delta_n\} \in \Delta$.

LEMMA 3.3. *Let $\{\delta_n\} \in \Delta$. For each fixed $\alpha > 0$ and $\varepsilon > 0$ the condition*

$$\lim_{n\to\infty} \int_{|x|\ge\varepsilon} (e^{\alpha|x|} - 1)|\delta_n(x)|\, dx = 0. \tag{4}$$

*is equivalent to*

$$\lim_{n\to\infty} \int_{\mathbb{R}} (e^{\alpha|x|} - 1)|\delta_n(x)|\, dx = 0. \tag{5}$$

*Proof.* It suffices to prove that (4) implies (5) for each fixed $\alpha > 0$ and $\varepsilon > 0$. For $|x| < \eta$, $(e^{\alpha|x|} - 1)$ is less than $(e^{\alpha\eta} - 1)$ which tends to zero as $\eta \to 0$. Hence choose $\eta$ such that $(e^{\alpha|x|} - 1) \le \varepsilon/2M$ with $M$ as in ($\Delta2$).

Hence

$$\int_{\mathbb{R}}(e^{\alpha|x|}-1)|\delta_n(x)|\,dx = \int_{|x|<\eta}(e^{\alpha|x|}-1)|\delta_n(x)|\,dx + \int_{|x|\geq\eta}(e^{\alpha|x|}-1)|\delta_n(x)|\,dx.$$

The first integral in the right hand side of the above equality is less than $\varepsilon/2$ by the choice of $\eta$ and the second integral tends to zero as $n$ tends to $\infty$ by hypothesis. ∎

LEMMA 3.4. *For $f \in \mathscr{G}'$ and $\{\sigma_n\} \in \hat{\Delta}$, $\sigma_n f \to f$ as $n \to \infty$ "strongly" in $\mathscr{G}'$ in the sense that for some $\beta > 0$,*

$$|\langle(\sigma_n f - f), \phi\rangle| \leq C_n \|\phi\|_\beta,$$

*where $C_n \to 0$ as $n \to \infty$. (Note that this in particular implies $\sigma_n f$ tends to $f$ in the weak* sense also.)*

*Proof.* Now since $f \in \mathscr{G}'$, by Theorem 1.5 there are positive constants $c$ and $\alpha$ such that for all $\phi \in \mathscr{G}$,

$$|f(\phi)| \leq c\|\phi\|_\alpha. \tag{6}$$

So

$$\begin{aligned} |(\sigma_n f - f)(\phi)| &= |f(\sigma_n \phi - \phi)| \\ &\leq c\|\sigma_n\phi - \phi\|_\alpha. \end{aligned} \tag{7}$$

Denoting the constant function $1(z) = 1$ by 1 we have

$$\begin{aligned} \|\sigma_n\phi - \phi\|_\alpha &= \sup_{z\in B_\alpha} e^{\alpha|\mathrm{Re}\, z|}|(\sigma_n\phi - \phi)(z)| \\ &= \sup_{z\in B_\alpha} e^{\alpha|\mathrm{Re}\, z|}|\phi(z)|\,|(\sigma_n - 1)(z)| \\ &\leq \sup_{z\in B_\alpha} e^{\alpha|\mathrm{Re}\, z|}e^{-\beta|\mathrm{Re}\, z|}\|\phi\|_\beta|(\sigma_n - 1)(z)|, \end{aligned}$$

where $\beta$ is chosen such that $\beta > \alpha$. Hence from (7) we have

$$\begin{aligned} |(\sigma_n f - f)(\phi)| &\leq c\|\phi\|_\beta \sup_{z\in B_\alpha} e^{(\alpha-\beta)|\mathrm{Re}\, z|}|(\sigma_n - 1)(z)| \\ &\leq c\, c_n\|\phi\|_\beta, \end{aligned} \tag{8}$$

where $c_n = \sup_{z\in B_\alpha} e^{(\alpha-\beta)|\mathrm{Re}\, z|}|(\sigma_n - 1)(z)|$.

We shall now prove that $c_n \to 0$ as $n \to \infty$. Using the properties of elements of $\Delta$ we get if $\hat{\delta}_n = \sigma_n$ (as $\{\sigma_n\} \in \hat{\Delta}$)

$$c_n = \sup_{z \in B_\alpha} e^{(\alpha - \beta)|\mathrm{Re}\, z|} \left| \left( \hat{\delta}_n - 1 \right)(z) \right|$$
$$\leq \frac{1}{\sqrt{2\pi}} \sup_{z \in B_\alpha} e^{(\alpha - \beta)|\mathrm{Re}\, z|} \int_{\mathbb{R}} |\delta_n(t)|\, |e^{-itz} - 1|\, dt. \tag{9}$$

Let $R$ be such that for a given $\varepsilon > 0$,

$$e^{(\alpha - \beta)R} < \frac{\varepsilon}{3M}, \tag{10}$$

where $M$ satisfies ($\Delta 2$). Let $A$ and $B$ be the subsets of $B_\alpha$ consisting of all $z$ such that $|\mathrm{Re}\, z| \leq R$ and $|\mathrm{Re}\, z| > R$, respectively. Then in the compact set $A, e^z$ is uniformly continuous. Hence given $\varepsilon/2M$ there exists $\eta > 0$, such that

$$|e^{-itz} - 1| < \frac{\varepsilon}{2M} \qquad \text{whenever } |t| < \eta. \tag{11}$$

Now since $B_\alpha$ is the disjoint union of $A$ and $B$, it suffices to show that the supremum over both $A$ and $B$ tends to 0 as $n \to \infty$. Put

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{|t| < \eta} |\delta_n(t)|\, |e^{-itz} - 1|\, dt$$
$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{|t| \geq \eta} |\delta_n(t)|\, |e^{-itz} - 1|\, dt.$$

In view of (11) and ($\Delta 2$)

$$I_1 < \frac{\varepsilon}{2}. \tag{12}$$

For $z \in A$ and $|t| \geq \eta$, it can easily be verified that

$$|e^{-itz} - 1| \leq (e^{\alpha|t|} + 1) \leq K(e^{\rho|t|} - 1),$$

where $K = 2/(1 - e^{-\rho\eta})$ and $\rho > \alpha$.

Thus by ($\Delta 3$), for large $n$,

$$I_2 < \frac{\varepsilon}{2}. \tag{13}$$

By (12) and (13) and the fact that $\alpha < \beta$ we have

$$\sup_{z \in A} e^{(\alpha-\beta)|\mathrm{Re}\, z|} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\delta_n(t)|\,|e^{-itz} - 1|\, dt < \varepsilon. \tag{14}$$

Take $z \in B$. Then for sufficiently large $n$

$$\begin{aligned} &e^{(\alpha-\beta)|\mathrm{Re}\, z|} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\delta_n(t)|\,|e^{-itz} - 1|\, dt \\ &\quad \le e^{(\alpha-\beta)R} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\delta_n(t)|\,|e^{-itz} - 1|\, dt \\ &\quad \le e^{(\alpha-\beta)R} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\delta_n(t)|(e^{\alpha|t|} + 1)\, dt \\ &\quad \le \frac{\varepsilon}{3M}(M + 2M) = \varepsilon. \end{aligned} \tag{15}$$

The last inequality is obtained using Lemma 3.3 and $(\Delta 2)$ and by writing $e^{\alpha|t|} + 1$ as $e^{\alpha|t|} - 1 + 2$. From (9), (14), and (15) we get for large $n$,

$$c_n = \sup_{z \in B_\alpha} e^{(\alpha-\beta)|\mathrm{Re}\, z|} |(\sigma_n - 1)(z)| < 2\varepsilon.$$

This proves that $c_n \to 0$ as $n \to \infty$. Hence we have from (8), $|(\sigma_n f - f)(\phi)| \le c\, c_n \|\phi\|_\beta$ where $c_n \to 0$ as $n \to \infty$. ∎

COROLLARY 3.5. *For $f \in \mathscr{G}'$, and $\{\delta_n\} \in \Delta$, $f * \delta_n \to f$ as $n \to \infty$ "strongly" in $\mathscr{G}'$ in the sense that for some $\rho > 0$,*

$$|\langle (f * \delta_n - f), \phi \rangle| \le D_n \|\phi\|_\rho,$$

*where $D_n \to 0$ as $n \to \infty$.*

*Proof.* Let $f \in \mathscr{G}'$ and $\phi \in \mathscr{G}$. Then we have

$$|\langle (f * \delta_n - f), \hat{\phi} \rangle| = \left| \left\langle \left( \hat{\delta}_n \hat{f} - \hat{f} \right), \phi \right\rangle \right| \le C_n \|\phi\|_\beta \le D_n \|\hat{\phi}\|_\rho.$$

The equality holds by virtue of the definition of the Fourier transform on $\mathscr{G}'$ and by Theorem 1.7(ii). The first inequality holds by Lemma 3.4 since $\{\hat{\delta}_n\} \in \hat{\Delta}$ and $\hat{f} \in \mathscr{G}'$. The last inequality holds by Theorem 1.3 for some $\rho > 0$ where $D_n$ is a constant multiple of $C_n$. Thus for all $\phi \in \mathscr{G}$

$$|\langle (f * \delta_n - f), \phi \rangle| \le D_n \|\phi\|_\rho, \text{ where } D_n \to 0 \text{ as } n \to \infty. \quad ∎$$

We shall state the following three lemmas without proofs. The proofs of Theorem 5.3, Lemma 6.2, and Theorem 3.2 of [3, Chap. VIII, pp. 184, 185, 182] can be easily adapted to prove Lemma 3.6, Lemma 3.7, and Lemma 3.8, respectively, once we note that the crucial integrability conditions involved therein remain valid in our case also when $F$ is assumed to have the following property:

$$F(w) \text{ is entire and } |F(u+iv)| \leq M e^{\alpha|u| + (\alpha/2)|v| + v^2/4}. \tag{P}$$

In the next three lemmas we assume that $F$ satisfies (P)

LEMMA 3.6. *If*

$$u(\zeta, t) = e^{-(1-t)D^2} F(\zeta) \qquad \textit{and}$$

$$v(\sigma, t+\delta) = \int_{\mathbb{R}} k(\sigma - \xi, t) u(\xi, \delta)\, d\xi$$

*then*

(i) $u(\sigma, t+\delta)$ *and* $v(\sigma, t+\delta)$, *as functions of* $(\sigma, t)$, *are in* $\mathscr{C}^{(2)}$ *and* $u_{\sigma\sigma} = u_t$ *and* $v_{\sigma\sigma} = v_t$ *in* $0 < t \leq c < 1$ *for every* $c < 1$.

(ii) $\lim_{\substack{\sigma \to \sigma_0 \\ t \to 0+}} (u(\sigma, t+\delta) - v(\sigma, t+\delta)) = 0$ *for all* $\sigma_0 \in \mathbb{R}$.

(iii) *If* $h_1(\sigma) = \max_{0 < t \leq c} |u(\sigma, t+\delta)|$ *and* $h_2(\sigma) = \max_{0 < t \leq c} |v(\sigma, t+\delta)|$ *then* $h_1(\sigma) = O(e^{\alpha\sigma^2})$ *and* $h_2(\sigma) = O(e^{b\sigma^2})$ *as* $|\sigma| \to \infty$ *for some* $a$ *and* $b$.

LEMMA 3.7. *For* $0 < \delta < c$, $0 < t < c - \delta$, $c < 1$, *and* $-\infty < \sigma < \infty$,

$$u(\sigma, t+\delta) = \int_{\mathbb{R}} k(\sigma - \xi, t) u(\xi, \delta)\, d\xi.$$

LEMMA 3.8. $u(\sigma, t) = \int_{\mathbb{R}} k(\eta, t) F(\sigma + i\eta)\, d\eta$ *and* $u(w, 1-) = F(w)$ *for all* $w \in \mathbb{C}$.

THEOREM 3.9 (Characterization of the Weierstrass transform for elements of $\mathscr{G}'$). *The conditions*

(I) *$F$ has property* (P)

(II) $|\langle e^{-tD^2} F, \phi \rangle| \leq M \|\phi\|_\alpha$ $(0 < t < 1, \phi \in \mathscr{G})$

*for some* $\alpha > 0$, *where, using the notation of* [3, *p.* 179],

$$e^{-tD^2} F(w) = \int_{\mathbb{R}} k(\xi + iw, t) F(i\xi)\, d\xi$$

*are necessary and sufficient conditions that* $F(w) = \tilde{f}(w)$ *for some* $f \in \mathscr{G}'$.

*Proof.* We first prove that conditions (I) and (II) are necessary. If $F(w) = \tilde{f}(w)$ for some $f \in \mathscr{G}'$, then we have with $w = u + iv$,

$$
\begin{aligned}
|F(u+iv)| &= |\langle f(z), k(w-z,1)\rangle| \\
&= \left| \frac{e^{-w^2/4}}{\sqrt{4\pi}} \langle f(z), \bar{e}^{(z^2-2wz)/4}\rangle \right| \\
&\le M_1 \bar{e}^{(u^2-v^2)/4} | |\bar{e}^{(z^2-2wz)/4}| |_\alpha \qquad \text{(by Theorem 1.5)} \\
&\le M e^{\alpha|u|+(\alpha/2)|v|+v^2/4} \qquad \text{by routine calculations.}
\end{aligned}
$$

If $f \in \mathscr{G}'$ and $f_n = f * \delta_n$ for some $\{\delta_n\} \in \Delta$ then by Corollary 3.5, there exists $\beta > 0$ such that

$$|\langle (f_n - f), \phi\rangle| = |\langle (f * \delta_n - f), \phi\rangle| \le D_n \|\phi\|_\beta,$$

where $D_n \to 0$ as $n \to \infty$. Thus, by the triangle inequality, for large $n$ and some $\rho > 0$

$$|\langle f_n, \phi\rangle| \le M\|\phi\|_\rho \qquad (\phi \in \mathscr{G}). \tag{16}$$

By an application of Fubini's theorem we get

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} f_n(x) k(\xi + iu, t) k(i\xi - x, 1)\, d\xi dx \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} f_n(x) k(\xi + iu, t) k(i\xi - x, 1)\, dx d\xi.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\left\langle f_n(z), \int_{\mathbb{R}} k(\xi + iu, t) k(i\xi - z, 1)\, d\xi \right\rangle \\
&\quad = \int_{\mathbb{R}} \langle f_n(z), k(\xi + iu, t) k(i\xi - z, 1)\rangle\, d\xi.
\end{aligned}
$$

Allowing $n \to \infty$ we see that (as a consequence of the Lebesgue's dominated convergence theorem which is seen to be valid using (16))

$$
\begin{aligned}
&\left\langle f(z), \int_{\mathbb{R}} k(\xi + iu, t) k(i\xi - z, 1)\, d\xi \right\rangle \\
&\quad = \int_{\mathbb{R}} \langle f(z), k(\xi + iu, t) k(i\xi - z, 1)\rangle\, d\xi.
\end{aligned}
$$

Now,

$$\begin{aligned} e^{-tD^2}F(u) &= \int_{\mathbb{R}} k(\xi + iu, t) F(i\xi)\, d\xi \\ &= \int_{\mathbb{R}} k(\xi + iu, t) \langle f(z), k(i\xi - z, 1)\rangle\, d\xi \\ &= \int_{\mathbb{R}} \langle f(z), k(\xi + iu, t) k(i\xi - z, 1)\rangle\, d\xi \\ &= \left\langle f(z), \int_{\mathbb{R}} k(\xi + iu, t) k(i\xi - z, 1)\, d\xi \right\rangle \\ &= \langle f(z), k(u - z, 1 - t)\rangle \in \mathscr{G}'. \end{aligned}$$

In obtaining the last inequality we have used Theorem 2.4 of [3, Chap. VIII, p. 177].

Therefore $e^{-tD^2}F(w) = \sqrt{2\pi}(f * k_{1-t})(w)$ for $w \in \mathbb{C}$ by the principle of analytic continuation. Thus, if $\phi \in \mathscr{G}$,

$$\begin{aligned} \left|\left(e^{-tD^2}F - f\right)\left(\check{\phi}\right)\right| &= \left|\left(\sqrt{2\pi}\,(f * k_{1-t}) - f\right)\left(\check{\phi}\right)\right| \\ &= \left|\left(\sqrt{2\pi}\,(f * k_{1-t}) * \phi\right)(0) - f\left(\check{\phi}\right)\right| \\ &= \left|\sqrt{2\pi}\,(f * (k_{1-t} * \phi))(0) - f\left(\check{\phi}\right)\right| \\ &= \left| f\left(\sqrt{2\pi}\,(k_{1-t} * \phi)\check{\ }\right) - f\left(\check{\phi}\right)\right| \\ &= \left| f\left(\sqrt{2\pi}\,(k_{1-t} * \phi)\check{\ } - \check{\phi}\right)\right| \\ &\leq M_1 \left\|\sqrt{2\pi}\,(k_{1-t} * \phi)\check{\ } - \check{\phi}\right\|_{\beta} \\ &= M_1 \left\|\sqrt{2\pi}\,(k_{1-t} * \phi) - \phi\right\|_{\beta} \\ &\leq M_2 \left\|\sqrt{2\pi}\,(k_{1-t} * \phi)\hat{\ } - \hat{\phi}\right\|_{\beta+\rho} \\ &\leq M_2 \|\hat{\phi}\|_{\beta+\rho} \sup_{z \in B_{\beta+\rho}} |e^{-(1-t)z^2} - 1| \\ &\leq M_t \|\phi\|_{\beta+\rho+\lambda} = M_t \|\check{\phi}\|_{\mu} \text{ (say)} \end{aligned} \tag{17}$$

for some $\beta$, $\rho$, and $\lambda$. In the above series of inequalities we have used the continuity of the Fourier transform and its inverse from $\mathscr{G}$ onto $\mathscr{G}$. We observe that for $z \in B_{\beta+\rho'}$

$$|e^{-(1-t)z^2} - 1| \leq M\left(e^{-(1-t)x^2} + 1\right) \leq 2M$$

and so $M_t = \sup_{z \in B_{\beta+\rho}} |e^{-(1-t)z^2} - 1|$ is bounded. Hence from (17) we obtain

$$\left|\left(\sqrt{2\pi}\, e^{-tD^2}F\right)(\phi)\right| \le M_t \|\phi\|_\mu + |f(\phi)| \le M_4 \|\phi\|_\alpha .$$

Therefore for some $\alpha > 0$ and for all $\phi \in \mathscr{G}$,

$$|(e^{-tD^2}F)(\phi)| \le M \|\phi\|_\alpha$$

which proves (II).

Conversely suppose that (I) and (II) hold. Put

$$V = \{\phi \in \mathscr{G} : M\|\phi\|_\alpha < 1\} \qquad \text{and}$$
$$W = \{f \in \mathscr{G}' : |\langle f, \phi\rangle| \le 1 \ \forall \phi \in V\}.$$

Then by the Banach Alaoglu theorem [11, 3.15, p. 66], $W$ is weak* compact. Let $\mathbb{F} = \{e^{-t_n D^2}F\}$ where $\{t_n\}$ is any sequence tending to $1-$ as $n$ tends to $\infty$. Then (II) shows that $\mathbb{F} \subset W$. Let us consider two cases:

*Case* 1. There are infinitely many $n$ say $n_j$ $(j = 1, 2, \ldots)$ such that $e^{-s_j D^2}F$ (where we denote $t_{n_j}$ by $s_j$) as a constant in $\mathscr{G}'$. In this case take this constant as $f$ and we have a sequence $e^{-s_j D^2}F$ converging to $f$ in $\mathscr{G}'$.

*Case* 2. $\mathbb{F}$ is an infinite set. Being an infinite subset of a compact set in the weak* topology of $\mathscr{G}'$, $\mathbb{F}$ has a limit point in the weak* topology of $\mathscr{G}'$, say $f$, by the Bolzano Weierstrass property. That is, every weak* neighborhood of $f$ contains an element of $\mathbb{F}$. Fix $\phi \in \mathscr{G}$. Then a typical weak* neighborhood of $f$ is of the form $N_U(\phi) = \{g \in \mathscr{G}' : g(\phi) \in U\}$ where $U$ is any neighborhood of the complex number $f(\phi)$ (see [11]). For $j = 1, 2, \ldots$ let $U_j = \{w : |w - f(\phi)| < 1/j\}$. Then there exists an $n_1$ such that $e^{-s_1 D^2}F \in N_{U_1}(\phi)$ where $t_{n_1}$ is denoted by $s_1$ and there exists an $n_2 > n_1$ such that $e^{-s_2 D^2}F \in N_{U_2}(\phi)$ where $t_{n_2}$ is denoted by $s_2$ and so on. Hence there exists a subsequence $\{s_j\} = \{t_{n_j}\}$ of $\{t_n\}$ such that $e^{-s_j D^2}F \in N_{u_j}(\phi)$. Hence as $j \to \infty$, $e^{-s_j D^2}F(\phi) \to f(\phi)$.

Hence in both cases we are able to find an $f \in \mathscr{G}'$ such that corresponding to each $\phi \in \mathscr{G}$ there exists a sequence $\{s_j\} \to 1-$ with $e^{-s_j D^2}F(\phi) \to f(\phi)$ as $j \to \infty$. Note that for different $\phi$ the sequence $\{s_j\}$ is in general different.

Let us now show that $F = \tilde{f}$ for this $f \in \mathscr{G}'$. Fix $t$ in the open line segment $(0, 1)$ and $\sigma \in \mathbb{R}$. Put $\phi(z) = k(\sigma - z, t)$. Now there exists a special sequence $\{s_j\}$ such that $s_j \to 1-$ and $e^{-s_j D^2}F(\phi) \to f(\phi)$ as $j \to \infty$.

Using our standard notation used in Lemma 3.6 and putting $s_j = 1 - \delta_j$ (so that $\delta_j$ tends to $0+$ as $j$ tends to $\infty$) we get from Lemma 3.7,

$$u(\sigma, t + \delta_j) = \langle u(z, \delta_j), k(\sigma - z, t)\rangle. \tag{18}$$

Allowing $j \to \infty$ in (18) we obtain

$$u(\sigma, t) = \langle f(z), k(\sigma - z, t)\rangle. \tag{19}$$

Let us now observe that (19) is valid for all $t$ in $(0, 1)$ despite the fact that for different $t$'s in $(0, 1)$ the associated sequences $\{\delta_j\}$ are also corresponding different.

It is easy to check that for each $\sigma \in \mathbb{R}$, $k(\sigma - z, t)$ tends to $k(\sigma - z, 1)$ in $\mathscr{G}$ as $t \to 1-$ (it is easier to check that the Fourier transform of $k(\sigma - z, t)$ tends to that of $k(\sigma - z, 1)$ in $\mathscr{G}$ which will imply the required result by the continuity of inverse of the Fourier transform). Hence allowing $t \to 1-$ in (19) and using Lemma 3.8 we get

$$F(w) = \langle f(z), k(w - z, 1)\rangle = \tilde{f}(w). \quad \blacksquare$$

We remark that if for some $g \in \mathscr{G}'$ $F(w) = \tilde{g}(w)$ then by Corollary 2.7, $f = g$ in $\mathscr{G}'$ proving the uniqueness of $f$ as an element of $\mathscr{G}'$, in Theorem 3.9.

## 4. A COMPARATIVE STUDY

In this section we prove the denseness of $\mathscr{G}$ in $\mathscr{L}(\alpha, \beta)$ for $\alpha > 0$ and $\beta < 0$ thereby proving that $\mathscr{L}'(\alpha, \beta)$ is a sub-space of $\mathscr{G}'$. Thus the Weierstrass transform theory will be directly applicable to $\mathscr{L}'(\alpha, \beta)$ in contrast with the classical situation. In the following we shall freely make use of the notations and results from Zemanian [12].

THEOREM 4.1. *$\mathscr{G}$ is dense in $\mathscr{L}(\alpha, \beta)$ for $\alpha > 0$ and $\beta < 0$.*

*Proof.* By definition $\mathscr{L}(\alpha, \beta) = \bigcup \mathscr{L}_{a,b}$ where $a > \alpha > 0$ and $b < \beta < 0$. It is easily verified that $\mathscr{G} \subset \mathscr{L}_{a,b}$ for all $a > \alpha > 0$ and $b < \beta < 0$ and that $\mathscr{L}(\alpha, \beta) \subset \mathscr{S}$, the space of rapidly decreasing functions. We shall use the sequence $\{\vartheta_\lambda(\phi * \omega_n)\}$ provided to us by Kenneth B. Howell in a private communication where $\vartheta_\lambda(z) = e^{-\lambda^2 z^2}$, $\omega_n(z) = ne^{-n^2 z^2/2}$ for $n \in \mathbb{N}$ and $\lambda > 0$ with $\vartheta(z) = e^{-z^2}$ and $\omega(z) = e^{-z^2/2}$. Let us first show that if $\psi \in \mathscr{L}(\alpha, \beta)$, then $\vartheta_\lambda \psi \to \psi$ in $\mathscr{L}(\alpha, \beta)$ as $\lambda \to 0$. Let $\psi \in \mathscr{L}_{c,d}$ for some $c$ and $d$ such that $\alpha < a < c$ and $d < b < \beta$. Then $\psi \in \mathscr{L}_{a,b}$ and we prove that $\vartheta_\lambda \psi \to \psi$ in $\mathscr{L}_{a,b}$ as $\lambda \to 0$. Let us show that, for simplicity, for the semi-norm $\gamma_1$, $\gamma_1(\vartheta_\lambda\psi - \psi) \to 0$ as $\lambda \to 0$. The cases for other semi-norms $\gamma_k$, $k = 2, 3\ldots$, follow similarly. By definition

$$\begin{aligned}
\gamma_1(\vartheta_\lambda\psi - \psi) &= \sup_{t\in\mathbb{R}}\left| x_{a,b}(t)\frac{d}{dt}\left[(\vartheta_\lambda(t) - 1)\psi(t)\right]\right| \\
&= \sup_{t\in\mathbb{R}}\left| x_{a,b}(t)\left[\psi'(t)\left[\vartheta_\lambda(t) - \vartheta(0)\right] + \psi(t)\vartheta'(\lambda t)\lambda\right]\right| \\
&= \lambda\sup_{t\in\mathbb{R}}\left| x_{a,b}(t)\left[\psi'(t)t\vartheta'(t_0)\right]\right| \\
&\quad + \lambda\sup_{t\in\mathbb{R}}\left| x_{a,b}(t)\psi(t)\vartheta'(\lambda t)\right|
\end{aligned}$$

for some $t_0$ between 0 and $\lambda t$ by mean value theorem. Let us observe that $\vartheta(z) \in \mathscr{G}$ and hence $|\vartheta'(x)| \le C$ for every $x \in \mathbb{R}$ and $t\psi' \in \mathscr{L}_{a,b}$ and hence $\lambda_1(\vartheta_\lambda\psi - \psi) \le M\lambda$ which tends to zero as $\lambda \to 0$. hence, $\vartheta_\lambda\psi \to \psi$ in $\mathscr{L}(\alpha, \beta)$.

Next we show that $\omega_n * \phi \to \phi$ in $\mathscr{L}(\alpha, \beta)$ for $\phi \in \mathscr{L}(\alpha, \beta)$. Let $\phi \in \mathscr{L}_{c,d}$. Then $\gamma_k(\omega_n * \phi - \phi)$ is given by

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\left| x_{c,d}(t)(\omega_n * \phi)^{(k)}(t) - \phi^{(k)}(t)\right| \\
&\quad = \sup_{t\in\mathbb{R}}\left| x_{c,d}(t)\left(\omega_n * (\phi)^{(k)}\right)(t) - \phi^{(k)}(t)\right| \\
&\quad = \frac{n}{\sqrt{2\pi}}\sup_{t\in\mathbb{R}}\left|\int_{\mathbb{R}} x_{c,d}(t)\left[(\phi)^{(k)}(\xi) - \phi^{(k)}(t)\right]e^{-n^2(t-\xi)^2/2}\,d\xi\right| \\
&\qquad\qquad \left(\text{using the fact that } \frac{n}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-n^2(t-\xi)^2/2}\,d\xi = 1\right) \\
&\quad \le \frac{n}{\sqrt{2\pi}}\sup_{t\in\mathbb{R}}\int_{\mathbb{R}} x_{c,d}(t)|t-\xi|\left|(\phi)^{(k+1)}(t_0)\right|e^{-n^2(t-\xi)^2/2}\,d\xi \\
&\qquad\qquad (\text{for some } t_0 \text{ in the open line segment joining } t \text{ and } \xi) \\
&\quad \le \frac{n}{\sqrt{2\pi}}\gamma_{k+1}(\phi)\sup_{t\in\mathbb{R}}\int_{\mathbb{R}} x_{c,d}(t)x_{c,d}^{-1}(t_0)|t-\xi|e^{-n^2(t-\xi)^2/2}\,d\xi.
\end{aligned}$$

Now we claim that for $t_0$ in the open line segment joining $t$ and $\xi$ where $t, \xi \in \mathbb{R}$,

$$\Re(t, t_0) = x_{c,d}(t)x_{c,d}^{-1}(t_0) \le e^{B|t-\xi|}.$$

In fact keeping in mind that $t_0$ lies in the open line segment joining $t$ and $\xi$, and that $c > 0 > d$

$$t \geq 0 \text{ and } t_0 \geq 0 \Rightarrow \Re(t, t_0) = e^{ct - ct_0} \leq e^{|c||t-\xi|}$$
$$t \leq 0 \text{ and } t_0 \leq 0 \Rightarrow \Re(t, t_0) = e^{dt - dt_0} \leq e^{|d||t-\xi|}$$
$$t \geq 0 \text{ and } t_0 \leq 0 \Rightarrow \Re(t, t_0) = e^{ct - dt_0} \leq e^{ct} \leq e^{ct - c\xi} \leq e^{|c||t-\xi|}$$
$$t \leq 0 \text{ and } t_0 \geq 0 \Rightarrow \Re(t, t_0) = e^{dt - ct_0} \leq e^{-d't + d'\xi} \leq e^{|d||t-\xi|}.$$

In the third combination we have assumed that $\xi \leq 0$ since $\xi > 0$ and $t \geq 0$ together will imply that $t_0 \geq 0$ which reduces to the first combination. Similarly in the fourth combination we have assumed that $\xi > 0$.

Thus in all combinations we see that $\Re(t, t_0) \leq e^{B|t-\xi|}$ where $B = \max\{|c|, |d|\}$. Therefore,

$$\begin{aligned}
\gamma_k(\omega_n * \phi - \phi) &\leq \frac{n}{\sqrt{2\pi}} \gamma_{k+1}(\phi) \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} |t - \xi| e^{B|t-\xi|} e^{-n^2(t-\xi)^2/2} \, d\xi \\
&= \frac{n}{\sqrt{2\pi}} \gamma_{k+1}(\phi) \sup_{t \in \mathbb{R}} \left(\frac{1}{n}\right)^2 \int_{\mathbb{R}} |\eta| e^{B|\eta|} e^{-\eta^2/2} \, d\eta \\
&\leq \frac{M}{n} \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

In the above calculations the equality is obtained by a change of variable. Hence $\phi \in \mathscr{L}_{c,d} \Rightarrow \phi * \omega_n \to \phi$ in $\mathscr{L}_{c,d} \Rightarrow \phi * \omega_n \to \phi$ in $\mathscr{L}_{a,b}$ and $\phi * \omega_n \in \mathscr{L}_{c,d} \Rightarrow \vartheta_\lambda(\phi * \omega_n) \to \phi * \omega_n$ in $\mathscr{L}_{a,b}$. Hence $\vartheta_\lambda(\phi * \omega_n) \to \phi$ in $\mathscr{L}_{a,b}$ as $\lambda \to 0$ and $n \to \infty$. One can easily verify that $\phi \in \mathscr{L}(\alpha, \beta)$ implies $\vartheta_\lambda(\phi * \omega_n) \in \mathscr{G}$ $\forall\ \lambda, n$ and we have shown that $\vartheta_\lambda(\phi * \omega_n) \to \phi$ in $\mathscr{L}(\alpha, \beta)$. This proves the theorem. ∎

Now Theorem 1.9.1 of [12, p. 24] implies the following corollary:

COROLLARY 4.2. *$\mathscr{L}'(\alpha, \beta)$ is a sub-space of $\mathscr{G}'$.*

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